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GREEN'S FUNCTIONS FOR SMALL PERTURBATIONS OF A PLANE INCOMPRESSIBLE COUETTE FLOW

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Formulas are obtained for the evolution of small three-dimensional perturbations arbitrarily specified at the initial instant of time of a plane Couette flow of incompressible viscous fluid in an unbounded space.

The behavior of small three-dimensional perturbations of the velocity $\mathbf{v}(\mathbf{r}, t)$ and pressure $p(\mathbf{r}, t)$ fields in an unbounded incompressible viscous fluid on the background of a plane Couette flow

$$u_x = u + \Gamma y, \quad u_y = u_z = 0, \quad P = \text{const} \tag{1}$$

where u and Γ are constants, was first considered in [1]. Formula for one of the components of velocity perturbations was also obtained there for some special form of initial conditions. Formulas which define the perturbation evolution under arbitrary initial conditions $\mathbf{v}(\mathbf{r})$ and $p(\mathbf{r})$ are derived below.

We proceed from the linearized Navier-Stokes equations [2]

$$\frac{\partial \mathbf{v}}{\partial t} + (u + \Gamma y) \frac{\partial \mathbf{v}}{\partial x} + \mathbf{x}^\circ \Gamma v_y + \frac{1}{\rho} \nabla p = \nu \Delta \mathbf{v}, \quad \operatorname{div} \mathbf{v} = 0 \tag{2}$$

where $\mathbf{x}^\circ$ is the unit vector of the Ox-axis.

We pass to the Fourier representation in space variables, where we shall use $\mathbf{k}$ instead of $\mathbf{r}$. Using (2) for $\mathbf{v}(\mathbf{k}, t)$ and $p(\mathbf{k}, t)$ we obtain

$$\mathbf{v}(\mathbf{k}, t) = \mathbf{f}(\mathbf{k}, t) \exp[-ituk_x - \nu t Q(\mathbf{k}, t)] \tag{3}$$

$$p(\mathbf{k}, t) = \frac{k^{*4}}{k^4} p(\mathbf{k}^*) \exp[-ituk_x - \nu t Q(\mathbf{k}, t)]$$

$$Q(\mathbf{k}, t) = (1 + \Gamma^2 t^2/3) k_x^2 + \Gamma t k_x k_y + k_y^2 + k_z^2 \tag{4}$$

$$\mathbf{k}^* = (k_x^*, k_y^*, k_z^*) = (k_x, k_y + \Gamma k_x t, k_z) \tag{5}$$

$$f_x(\mathbf{k}, t) = v_x(\mathbf{k}^*) + \frac{k_x}{k_x^2 + k_z^2} \left(k_y^* - k_y \frac{k^{*2}}{k^2} \right) v_y(\mathbf{k}^*) + \tag{6}$$

$$+ \frac{k_z^2 k^{*2}}{k_x (k_x^2 + k_z^2)^{3/2}} \left[\operatorname{arctg} \left(\frac{k_y}{\sqrt{k_x^2 + k_z^2}} \right) - \operatorname{arctg} \left(\frac{k_y^*}{\sqrt{k_x^2 + k_z^2}} \right) \right] v_y(\mathbf{k}^*)$$

$$f_y(\mathbf{k}, t) = \frac{k^{*2}}{k^2} v_y(\mathbf{k}^*)$$

$$f_z(\mathbf{k}, t) = v_z(\mathbf{k}^*) + \frac{k_z k^{*2}}{k_x^2 + k_z^2} \left\{ \frac{k_y^*}{k^{*2}} - \frac{k_y}{k^2} + \frac{1}{\sqrt{k_x^2 + k_z^2}} \times \right.$$

$$\left. \left[\operatorname{arctg} \left(\frac{k_y^*}{\sqrt{k_x^2 + k_z^2}} \right) - \operatorname{arctg} \left(\frac{k_y}{\sqrt{k_x^2 + k_z^2}} \right) \right] \right\} v_y(\mathbf{k}^*)$$

We set in Eqs. (2)

$$\mathbf{v}(\mathbf{r}) = \mathrm{rot}\,(\boldsymbol{\alpha}\delta(\mathbf{r} - \mathbf{r}_0)), \quad \alpha^2 = 1$$

where $\boldsymbol{\alpha}$ is some constant unit vector and $\delta(\mathbf{r})$ is a three-dimensional Dirac function, and substitute everywhere $t = t_0$ for the initial instant, $t = 0$. These initial conditions yield the Fourier representation of the tensor of Green's function $G_{ab}(\mathbf{r} - \mathbf{r}_0, t - t_0)$ of the considered problem of perturbation propagation from the instantaneous source at point $\mathbf{r} = \mathbf{r}_0$ at instant $t = t_0$. The structure of that tensor is defined by

$$G_{ab}(\mathbf{k}, t - t_0) = i\theta(t - t_0) \begin{Vmatrix} g_{xx} & g_{xy} & 0 \\ 0 & g_{yy} & 0 \\ 0 & g_{zy} & g_{zz} \end{Vmatrix} \times \tag{7}$$
$$\exp\left[-i\mathbf{k}^*(\mathbf{r}_0 + \mathbf{u}(t - t_0)) - \nu(t - t_0)\, Q(\mathbf{k}, t - t_0)\right]$$

where $\mathbf{u} = (u, 0, 0)$ and $\theta(t)$ is a Heaviside function. The diagonal elements of tensor $g_{ab}(\mathbf{k}, t - t_0)$ are

$$g_{xx} = [\mathbf{k}^*\boldsymbol{\alpha}]_x, \quad g_{yy} = [\mathbf{k}^*\boldsymbol{\alpha}]_y \frac{k^{*2}}{k^2}, \quad g_{zz} = [\mathbf{k}^*\boldsymbol{\alpha}]_z$$
$$\mathbf{k}^* = (k_x, k_y + \Gamma(t - t_0) k_x, k_z)$$

Similar, although more cumbersome, expressions can be obtained for g_{xy} and g_{zy} with the use of (6).

The explicit expression for tensor $G_{ab}(\mathbf{r} - \mathbf{r}_0, t - t_0)$ in coordinate representation is complicated and cumbersome; the structure of all of its components is of the form

$$G_{ab}(\mathbf{r} - \mathbf{r}_0, t - t_0) = \frac{\theta(t - t_0)}{\sqrt{(4\pi\nu(t - t_0))^3 (1 + \tau^2/12)}} \times \tag{8}$$
$$M_{ab}(\mathbf{r} \mid \boldsymbol{\alpha}, \tau) \exp\left\{-\frac{1}{4\nu(t - t_0)} \times \right.$$
$$\left.\left[\left(\frac{x - x_0 - u(t - t_0) - \frac{1}{2}\tau(y - y_0)}{\sqrt{1 + \tau^2/12}}\right)^2 + (y - y_0)^2 + (z - z_0)^2\right]\right\}$$

where $M_{ab}(\mathbf{r} \mid \boldsymbol{\alpha}, \tau)$ are linear operators acting on the functions of space coordinates $\mathbf{r}$ and of parameters dependent on unit vector $\boldsymbol{\alpha}$ and time in terms of the dimensionless combination $\tau = \Gamma(t - t_0)$. The drift of perturbations and their distortion with time can be seen from [8].

Owing to the positive definiteness of the quadratic form $Q(\mathbf{k}, t)$ as implied by (4) and the property $\nu > 0$, it follows from (7) that for all components of $G_{ab}(\mathbf{k}, t - t_0)$, $G_{ab}(\mathbf{k}, t - t_0) \mid \to 0$ when $t - t_0 \to \infty$, for any u and Γ. According to (8) components $G_{ab}(\mathbf{r} - \mathbf{r}_0, t - t_0)$ have the same property. Thus we obtain the known property of the stability of Couette flow (1) with respect to arbitrary small perturbations in an unbounded space for any u and Γ.

The behavior of small perturbations in more complicated flows of incompressible viscous fluid in an unbounded space can be similarly investigated. For example, for the flow

$$u_x = \Gamma y, \quad u_y = -\Gamma x, \quad u_z = 0$$

which corresponds to a uniform rotation of the fluid as a whole about the axis Oz at an angular velocity Γ the evolution of perturbations is defined by formulas

$$\mathbf{v}(\mathbf{k}, t) = \mathbf{f}(\mathbf{k}, t) \exp(-\nu k^2 t), \quad p(\mathbf{k}, t) = 2i\rho\,\Gamma(k_x v_y - k_y v_x) / k^2 \tag{9}$$

The expression for $\mathbf{f}(\mathbf{k}, t)$ is fairly cumbersome and is not presented here. We only point out that for considerable t the behavior of $\mathbf{v}(\mathbf{k}, t)$ is determined by the second factor in the first formula of (9), i. e. the perturbations fade with time. Hence the rotation of a viscous incompressible fluid is stable with respect to such perturbations.

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FLOW AROUND A SPHERICAL DROP AT INTERMEDIATE REYNOLDS NUMBERS

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A solution of the Navier-Stokes equations for the flow of fluid in and outside a drop with conditions of matching at the interface is derived by the method of finite differences. Drag coefficients are determined in the range $(0.5 \leqslant \mathrm{Re} \leqslant 100)$ of Reynolds numbers for a solid sphere, a drop, and a small gas bubble. Vortex and velocity distribution at the drop boundary is determined.

The flow around a solid sphere in the intermediate range of Reynolds numbers had been thoroughly investigated [1]. Solutions for the problem of flow around a spherical drop are presented in [2, 3] for $\mathrm{Re} \ll 1$. The method of joining asymptotic expansions was used in [4] for obtaining a solution for small Re with allowance for inertia terms in the Navier-Stokes equations. In [5 — 7] solutions were derived for $\mathrm{Re} \gg 1$ in the boundary layer approximation (a detailed analysis of approximate solutions for low and high Re appeared in the survey paper [8]). The particular case of the drop of water in air, which is distinguished by the high ratio viscosities of the inner and outer media ($\mu \approx 56$), was investigated in [9] in the intermediate range of Reynolds numbers by the method of finite differences. It was shown there that for such μ the drag of the moving drop is virtually the same as that of the solid sphere. Here, the drag of the drop is investigated for $0 \leqslant \mu < \infty$ and $\mathrm{Re} \leqslant 100$.

The rectilinear uniform motion of a drop in a homogeneous mass force field is considered. The Weber number is assumed to be fairly low so that the drop virtually retains its spherical shape.

In a system of coordinates attached to the drop the motion is steady and axisymmetric up to $\mathrm{Re} \approx 100$, as in the case of a solid sphere [10].

With the coordinate origin located at the drop center and the polar axis directed downstream $(\theta = 0)$, the Navier-Stokes equations for the fluid flow in and outside the drop and the boundary conditions at the drop surface, expressed in terms of variables ψ (the stream function) and ζ (the vortex), are of the form